Solving LPN Using Large Covering Codes

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Outline

Post-quantum cryptography

Cryptography based on problems that are hard both for classical and quantum computers.

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Learning Parity with Noise falls in the code-based category.
We want to qualify how hard the LPN problem is.

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So $a + b + b = a$.

Also, $1-1 = 0 = 1+1$

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Learning Parity *without* Noise

$$\mathsf{s} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Learning Parity *without* Noise

Through the magic of *Gaussian elimination*

$$\mathsf{s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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We add some noise to the computations. We flip a bit using a biased coin (Bernoulli distribution) that gives head (1) with probability τ.

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Suddenly, finding $\mathbf{s}$ is hard.
Hardness related to *decoding random codes*.

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$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \\ \mathbf{a}_5 \\ \mathbf{a}_6 \end{pmatrix} \cdot \mathbf{s} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

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Familiar? LWE is the same problem over $\mathbb{Z}^q$.

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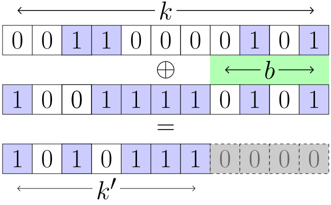

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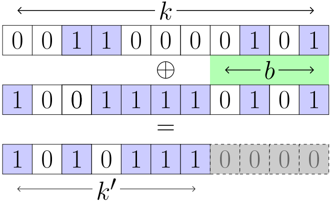

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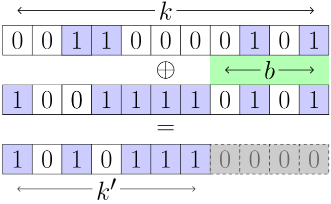

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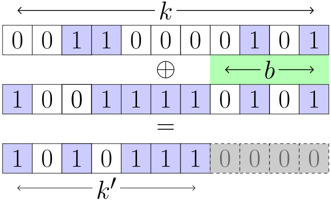

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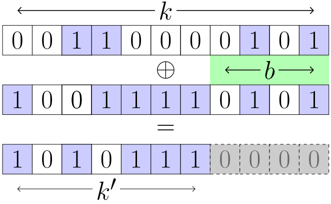

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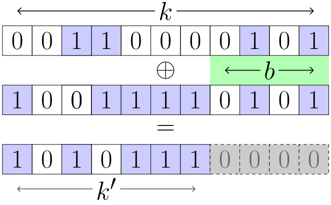

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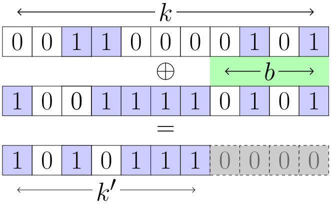

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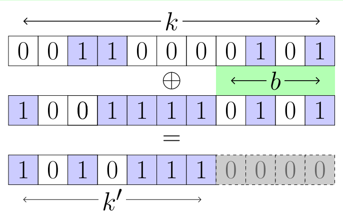

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Solve an $\mathrm{LPN}_{k,\tau}$ problem, given n samples.

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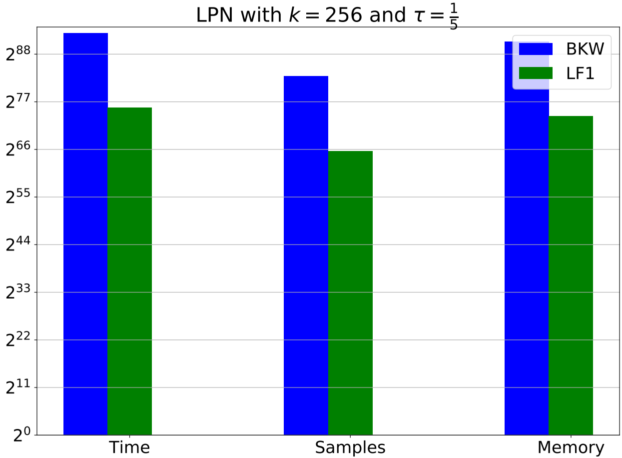
We may apply several reductions algorithms in sequence!

Complexity


LPN with $k = 256$ and $\tau = \frac{1}{5}$
2^{88}
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3. If $\mathbf{e}'$ has approximately $m \cdot \tau$ bits flipped, probably $\mathbf{s}' = \mathbf{s}$

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This algorithm is an *Information-Set Decoding* algorithm: it finds an error-free index set in the pool. Notably, it resembles the [Pra62] algorithm.

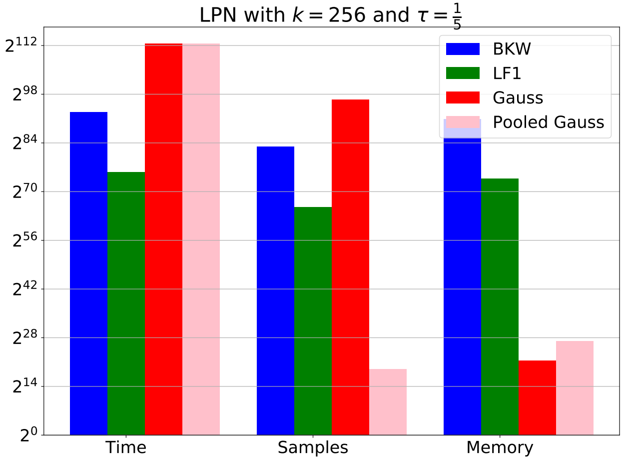
Complexities

To solve $\mathrm{LPN}_{k,\tau}$:

	n **Samples**	**Time**	**Memory**
BKW	$20 \cdot \ln(4k) \cdot 2^b \cdot (1-2\tau)^{-2^a}$	kan	kn
LF1	$(8b+2000)(1-2\tau)^{-2^a} + (a-1)2^b$	$kan + b2^b$	$kn + b2^b$
Gauss	$k \cdot I + m$	$\left(k^3 + km\right) I$	$k^2 + km$
Pooled Gauss	$k^2 \log^2 k + m$	$\left(k^3 + km\right) I$	$k^2 \log^2 k + km$

Gauss needs $I = O\left(\frac{\log_2^2 k}{(1-\tau)^k}\right)$ iterations to find a solution.

Complexity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Covering Codes

$$\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

$$\downarrow$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

Covering-codes reduction

$$
k' \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right. \leftarrow
\begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
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\end{bmatrix} \\
\uparrow \\
\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \\
\longleftarrow k \longrightarrow
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\uparrow\\
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\end{array}
$$

This allows us to reduce a k-size LPN problem with noise τ to a k'-sized LPN problem with noise τ'.
This new noise τ' is strongly dependent on the code used. We measure the impact as bc. ($0 \leq \text{bc} \leq 1$, larger is better)

Finding codes for the reduction

- We need a code that allows us to reduce from k to k'.

$$k' \left\{ \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right. \leftarrow \begin{bmatrix}1&0&0&0&0&1&1\\0&1&0&0&1&0&1\\0&0&1&0&1&1&0\\0&0&0&1&1&1&1\end{bmatrix}$$

$$\uparrow$$

$$\underbrace{\begin{bmatrix}0&1&0&1&1&0&1\end{bmatrix}}_{k}$$

Finding codes for the reduction

- We need a code that allows us to reduce from k to k'.
- We could use random codes, but they are hard to decode.

$$
\begin{array}{cccc}
k' \updownarrow & \begin{array}{|c|}\hline 0\\\hline 0\\\hline 1\\\hline 0\\\hline\end{array} & \leftarrow & \begin{array}{|c|c|c|c|c|c|c|}\hline 1&0&0&0&0&1&1\\\hline 0&1&0&0&1&0&1\\\hline 0&0&1&0&1&1&0\\\hline 0&0&0&1&1&1&1\\\hline\end{array} \\
 & & & \uparrow \\
 & & & \begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&0&1&1&0&1\\\hline\end{array} \\
 & & & \longleftarrow k \longrightarrow
\end{array}
$$

Finding codes for the reduction

- We need a code that allows us to reduce from k to k'.
- We could use random codes, but they are hard to decode.
- (Quasi-)Perfect codes give the best bc, but only few are known.

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{k'} \leftarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \uparrow \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}}_{k}$$

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$\rightarrow$ Time complexity $O(n + (k^3 + k \cdot m) \cdot I)$

How 'good' should a code be?

Let's assume we have arbitrary $[k, k']$ codes.

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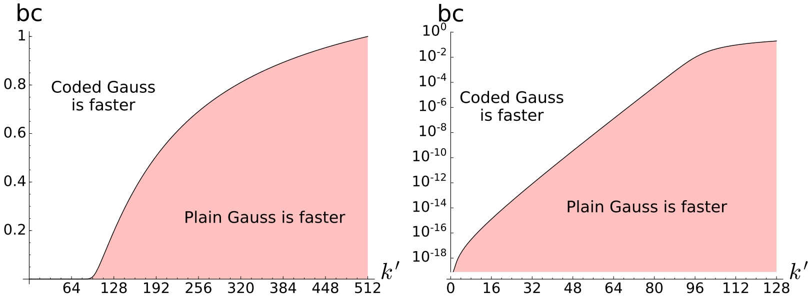
Let’s assume we have arbitrary $[k, k']$ codes. We have the following inequality

$$T_{\text{Gauss}}(k, \tau) \geq T_{\text{Coded Gauss}}(k, k', \tau, \text{bc})$$

How ‘good’ should a code be?

Let’s assume we have arbitrary $[k, k']$ codes.



(a) Lower bound for bc

(b) More detailed look at $0 < k' \leq 128$.

Figure: Minimal bc before Coded Gauss is faster than applying Gauss to the full problem. $k = 512, \tau = \frac{1}{8}$.

The best-case scenario for bc

Assume we have arbitrary, (quasi-)perfect codes.

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$$\mathrm{bc} \leq 2^{k'-k} \sum_{w=0}^{R} \binom{k}{w} \left(\delta_s^w - \delta_s^{R+1} \right) + \delta_s^{R+1}.$$

Here, R is a property we can bound for quasi-perfect codes (Hamming Bound [Ham50]) and $\delta_s = 1 - 2\tau$.

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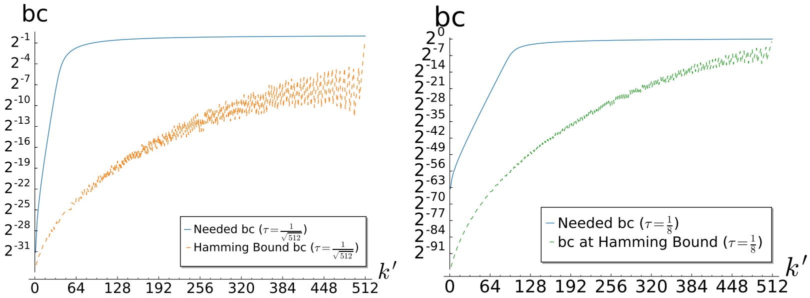

(a) $\tau = \frac{1}{\sqrt{512}}$ (b) $\tau = \frac{1}{8}$

Figure: Minimal bc and the bc obtained at the Hamming bound for various τ. $k = 512, \delta = \delta_s = 1 - 2\tau$.

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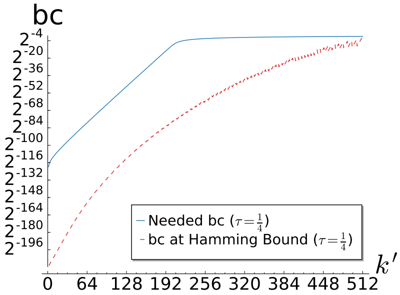

(a) $\tau = \frac{1}{4}$ (b) $\tau = \frac{4\,999\,999}{10\,000\,000}$

Figure: Minimal bc and the bc obtained at the Hamming bound for various τ. $k = 512, \delta = \delta_s = 1 - 2\tau$.

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In conclusion, the following:

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Note

Our analysis was limited to the above algorithm. We have results that show the following *may* work

1. Apply some reduction to reduce to a k'-sized problem with noise τ'
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However, we would also need to include the complexity of step 1 when analysing this combination.

Outline

Improving the performance of the covering-codes reduction

The covering-codes reduction as originally proposed:

1. Apply covering-codes reduction
2. Recover information on $\mathbf{s}$ using Walsh-Hadamard Transform.

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Picking codes is hard. Much of the work around this attack has been on finding the right codes to instantiate attacks.

Concatenated Codes

Current attacks use concatenations of small perfect codes to construct larger $[k, k']$ codes.

Example

We construct the following $[12, 4]$ code from $[3, 1]$ repetion codes with generator $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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The bc of concatenated codes is the product of the bc of the smaller codes.

Concatenated Codes (cont.)

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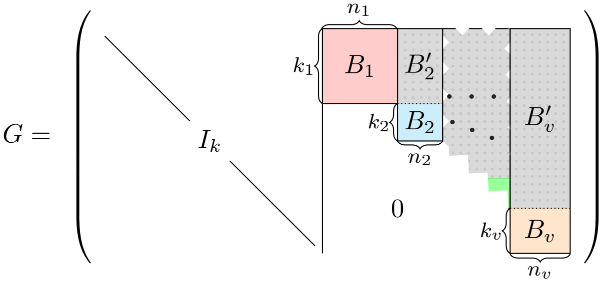
Decoding Algorithm

1. Generate look up tables for the small codes
2. Split your vector along the small codes
3. Look up the codewords for the individual pieces in the lookup tables
4. Concatenate

StGen codes

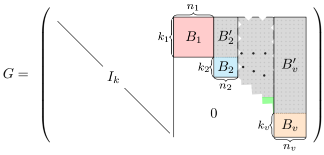
Samardjiska and Gliogoski proposed an improvement on these concatenations of codes. We add random noise on top of the blocks.



(1)

Simona proposed using these codes with the covering-codes reduction at a department lunch talk.

Decoding StGen Codes

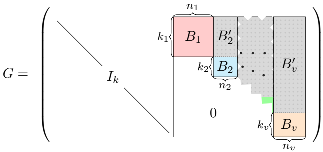


(2)

Decoding algorithm sketch

1. Set maximum error weights and limits

Decoding StGen Codes



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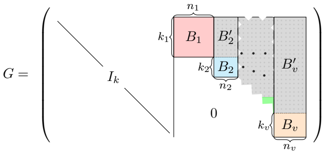

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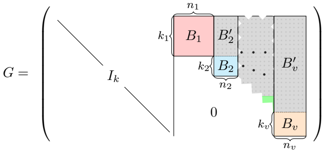

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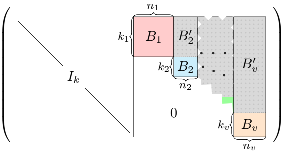
$G =$ 

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Decoding algorithm sketch

1. Set maximum error weights and limits
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5. Generate all the candidates for B_2

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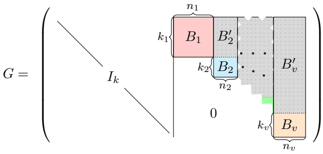

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6. Increase maximum weights if you have few candidates for the next round.

Complications of StGen codes

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- Because of the random elements we can no longer directly compute bc.
  - For random codes computing bc is a hugely expensive operation.
  - We instead estimate it over a number of random vectors

Outline

Finding reduction chains

Bogos and Vaudenay propose a search algorithm for finding combinations of reductions:

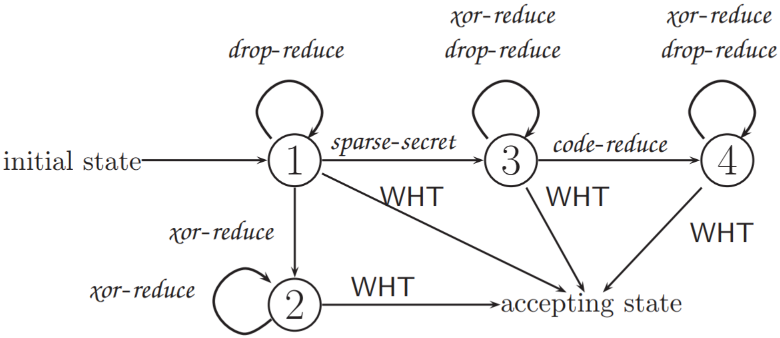

Figure: Finding chains of reductions [Bog17].

Improving the performance of an attack

Bogos and Vaudenay propose a combination of reductions to solve $\mathrm{LPN}_{512,\frac{1}{8}}$ in $\mathcal{O}(2^{78.85})$ time using $2^{63.3}$ samples.

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Step	k	$\log_2 n$	$1-2\tau$	δ_s	Algorithm
1	512	63.3	0.75	0	`sparse-secret`
2	512	63.3	0.75	0.75	`xor-reduce` ($b = 59$)
3	453	66.6	0.5625	0.75	`xor-reduce` ($b = 65$)
4	388	67.2	0.3164	0.75	`xor-reduce` ($b = 66$)
5	322	67.4	0.1001	0.75	`xor-reduce` ($b = 66$)
6	256	67.8	0.0100	0.75	`xor-reduce` ($b = 67$)
7	189	67.6	0.0001	0.75	`covering-codes`
8	64	67.6	$8.8 \cdot 10^{-10}$		`FWHT`

Table: The full solving chain of Bogos and Vaudenay [BV16; BV] on $\mathrm{LPN}_{512,\frac{1}{8}}$. In step 7 they apply a $[189, 64]$ covering code with $\mathrm{bc} = 8.78 \cdot 10^{-6}$.

The code used by Bogos and Vaudenay

The last reduction applied uses a number of random codes.

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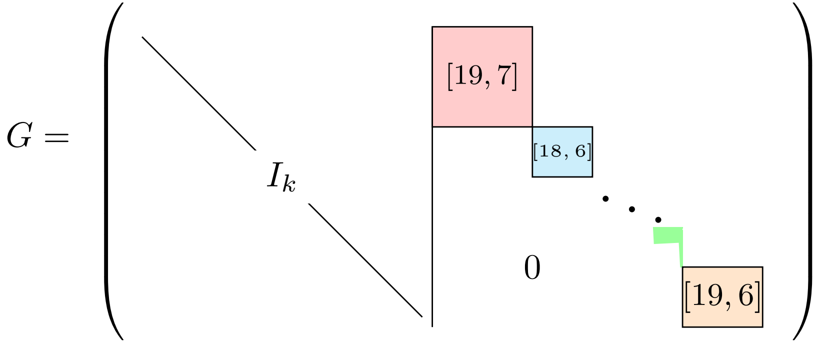
Table: bc for the small random codes used in the solving algorithm for $\mathrm{LPN}_{512,\frac{1}{8}}$ [BV16; BV].

Code	**Count**	**bc** $(\tau = \frac{1}{8})$
$[18, 6]$	1	0.323782920837402
$[19, 6]$	5	0.291754990816116
$[19, 7]$	4	0.336303114891052

The code used by Bogos and Vaudenay



The bc of the concatenated code is

$$\mathrm{bc} = 0.323^1 \cdot 0.292^5 \cdot 0.336^4 = 8.78 \cdot 10^{-6}.$$

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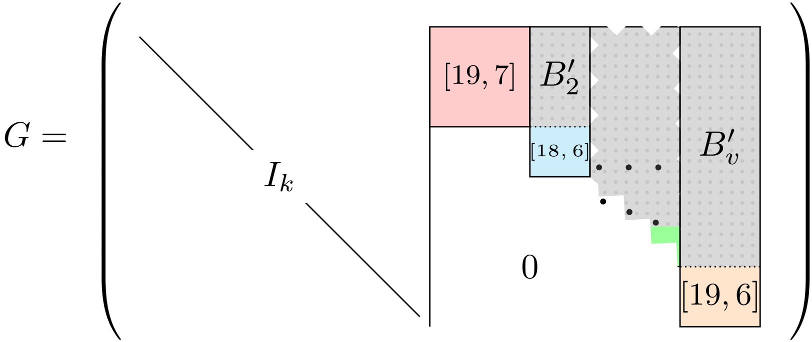

The bc of this StGen code is approximated to

$$\mathrm{bc} \approx 3.8 \cdot 10^{-5}.$$

Theoretical improvement

Using $\mathrm{bc} = 3.8 \cdot 10^{-5}$ we improve the performance of the algorithm.

Table: Improved attack on $LPN_{512,\frac{1}{8}}$

	Original	**With StGen code**
Time	$\mathcal{O}\left(2^{78.85}\right)$	$\mathcal{O}\left(2^{78.1}\right)$
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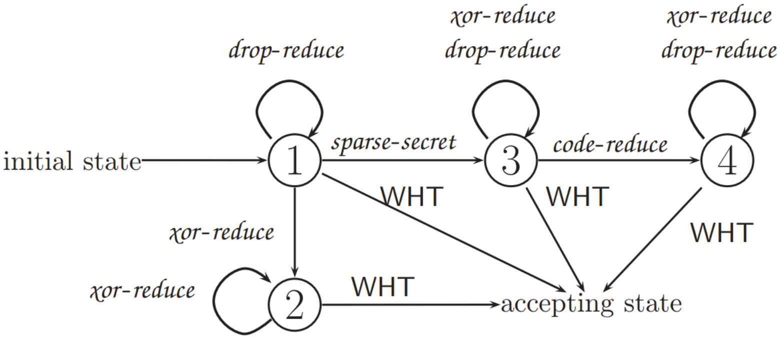

Figure: Finding chains of reductions [Bog17].

Finding new reduction chains

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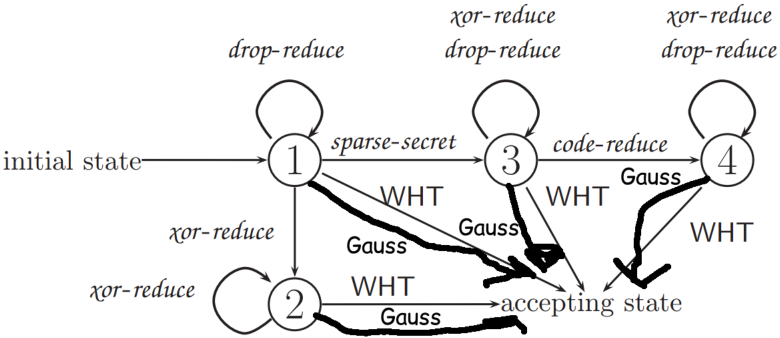

Figure: Finding chains of reductions with Gauss [Bog17].

Memory consumption of *m*

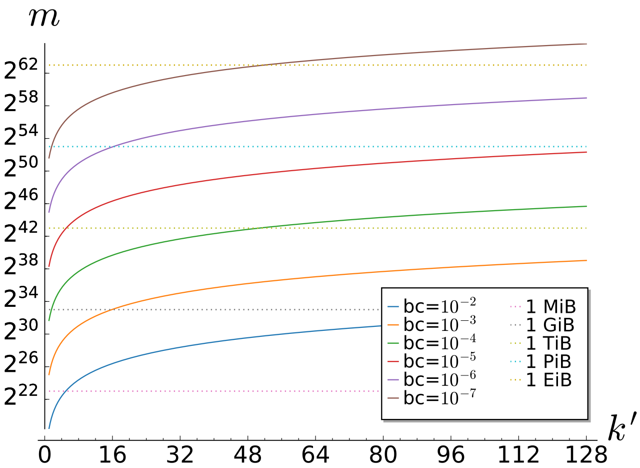

Figure: *m* for various small bc ($\tau = \frac{1}{8}$)

Software

We developed software that allows to implement LPN solving algorithms.

```
// Create LPN oracle with k=32 and tau=1/32
let mut oracle = LpnOracle::new(32, 1.0 / 32.0);
oracle.get_samples(1000);
// apply the LF2 `xor_reduce' reduction
// using b = 8 three times
xor_reduction(&mut oracle, 8);
xor_reduction(&mut oracle, 8);
xor_reduction(&mut oracle, 8);
// solve using two techniques
let fwht_solution = fwht_solve(oracle.clone());
let gauss_solution = pooled_gauss_solve(oracle);
```

Available via `https://thomwiggers.nl/research/msc-thesis/`.

Wrapping up

Conclusions

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Work in progress

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Thank you for your attention

Outline

The BKW algorithm

Input: A set V of n samples $(\mathbf{a}, c)$ from $\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},t}$, a, b s.t. $k \geq ab$

for $i = 1$ **to** $a - 1$ **do**

  // Reduction (partition-reduce):

  Partition $V = V_1 \cup \cdots \cup V_{2^b}$ s.t. they all have the same bit values on the last ib bits

  foreach V_j **do**

    Choose a $(\mathbf{a}', c') \in V_j$

    Replace all other $(\mathbf{a}, c) \in V_j$ by $(\mathbf{a} + \mathbf{a}', c + c')$

    Discard $(\mathbf{a}', c')$

// Solving phase (majority):

Discard all samples $(\mathbf{a}, c)$ from V where $HW(\mathbf{a}) \neq 1$

Divide V into b partitions, such that vectors $\mathbf{a} \in V_j$ have $\mathbf{a}_j = 1$

for $i = 1$ **to** b **do**

  $\mathsf{s}_i = \text{majority}(c)$, for all $(\mathbf{a}, c) \in V_i$

return $\mathsf{s}_1, \ldots, \mathsf{s}_b$

LF1 algorithm

Algorithm 1: The LF1 algorithm as presented in [BTV15]

Input: A set V of n samples $(\mathbf{a}, c)$ from $\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},t}$, a, b s.t. $k = ab$

Output: $(\mathbf{s}_1, \ldots, \mathbf{s}_a)$ from $\mathbf{s}$

Run $a - 1$ iterations of `partition-reduce` as in the BKW algorithm

// Solving Phase (FWHT):

$f(\mathbf{x}) = \sum_{(\mathbf{a},c)\in V} 1_{V_{1,\ldots,b}=\mathbf{x}}(-1)^c$

$\hat{f}(\mathbf{x}) = \sum_x (-1)^{\langle \mathbf{a},\mathbf{x}\rangle} f(x)$

return $(\mathbf{s}_1, \ldots, \mathbf{s}_b) = \arg\max_{\mathbf{a}\in\mathbb{Z}_2^b}(\hat{f}(\mathbf{a}))$

LF2 algorithm

Algorithm 2: The LF2 algorithm [LF06]

Input: A set V of n samples $(\mathbf{a}, c)$ from $\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},t}$,
a, b s.t. $k = ab$

Output: $(\mathbf{s}_1, \ldots, \mathbf{s}_b)$ from $\mathbf{s}$

for $i = 1$ **to** $a - 1$ **do**
  Partition $V = V_1 \cup \cdots \cup V_{2^b}$ s.t. they all have the same bit values on the last ib bits
  foreach V_j **do**
    $V'_j = \emptyset$
    for $(\mathbf{a}, c), (\mathbf{a}', c') \in V_j, (\mathbf{a}, c) \neq (\mathbf{a}', c')$ **do**
      $V'_j = V'_j \cup \{(\mathbf{a} + \mathbf{a}', c + c')\}$
  $V = V'_1 \cup \cdots \cup V'_{2^b}$

// Solving Phase (FWHT) [..]

return $(\mathbf{s}_1, \ldots, \mathbf{s}_b) = \arg\max(\hat{f}(\mathbf{a}))$

Gausss

Function `Gauss`$(\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},\tau}, \tau)$

  repeat

    repeat

      $(A, \mathbf{c}) \leftarrow \left(\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},\tau}\right)^k$

    until *A is full rank*

    $\mathbf{s}' = A^{-1}\mathbf{c}$

  until `Test`$(\mathbf{s}', \tau, \frac{1}{2^k}, \left(\frac{1-\tau}{2}\right)^k)$

  return $\mathbf{s}'$

Function `Test`$(\mathbf{s}', \tau, \alpha, \beta)$

  $m = \left(\frac{\sqrt{\frac{3}{2}\ln\left(\frac{1}{\alpha}\right)}+\sqrt{\ln\frac{1}{\beta}}}{\frac{1}{2}-\tau}\right)^2;$

  $c = \tau m + \sqrt{3\left(\frac{1}{2}-\tau\right)\ln\left(\frac{1}{\alpha}\right)m};$

  $(A, \mathbf{c}) \leftarrow \left(\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},\tau}\right)^m;$

  if $HW(A\mathbf{s}' + \mathbf{c}) \leq c$ **then**

    return *True*;

  else

    return *False*;

Pooled Gauss

Function $\texttt{PooledGauss}(\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},\tau}, \tau)$

  $P \leftarrow \left(\mathcal{O}^{\mathsf{LPN}}_{\mathbf{s},\tau}\right)^{k^2 \log_2 k}$

  repeat

    repeat

      $(A, \mathbf{c}) \xleftarrow{U} P$

    until *A is full rank*

    $\mathbf{s}' = A^{-1}\mathbf{c}$

  until $\texttt{Test}(\mathbf{s}', \tau, \frac{1}{2^k}, \left(\frac{1-\tau}{2}\right)^k)$

  return $\mathbf{s}'$

When is Coded Gauss faster

$$\frac{\left(k^3 + km\right)\log_2^2 k}{\left(\frac{1}{2} + \frac{1}{2}\delta\right)^k} \geq \frac{\left({k'}^3 + k'm\right)\log_2^2 k'}{\left(\frac{1}{2} + \frac{1}{2}\delta \mathrm{bc}\right)^{k'}} + m + n.$$

Memory consumption of *m*

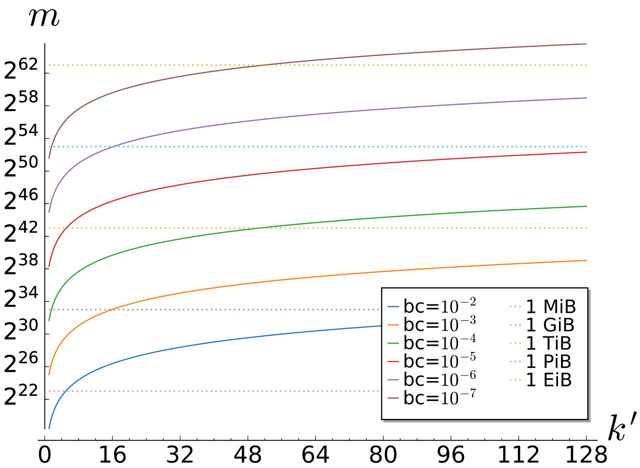

Figure: *m* for various small bc ($\tau = \frac{1}{8}$)

StGen decoding

Input: $w_1, w_b, w_{\text{inc}}, G, L_{\max}, \mathbf{c} \in \mathbb{F}_2^n$.
Output: A close codeword of $\mathbf{c}$
Let $K_i = \Sigma_{j=1}^{i} k_j$, $N_i = \Sigma_{j=1}^{i} n_j$ and let G_i be the 'small code' $(I_{k_i}|B_i)$.

```
L_0 = {(x_0, e_0)}, x_0, e_0 are zero-dimensional vectors.
for i = 1 to v do
    foreach (x_{i-1}, e_{i-1}) in L_{i-1} do
        b = (c_{K_{i-1}}, ..., c_{K_i}) || (x_{i-1} B'_i + (c_{k+N_{i-1}}, ..., c_{k+N_i}))
        max-wt = min(w_i - HW(e_{i-1}), w_b)
        foreach e' ∈ { v ∈ F_2^{n_i+k_i} | HW(v) ≤ max-wt } do
            Find x' s.t. x' G_i + b = e'
            e_new = ((e_{i-1})_1, ..., (e_{i-1})_{K_{i-1}}, e'_1, ..., e'_{k_i},
                     (e_{i-1})_{K_{i-1}}, ..., (e_{i-1})_{K_{i-1}+N_{i-1}}, e'_{k_i}, ..., e'_{k_i+n_i})
            Add (x_{i-1}||x', e_new) to L_i
    if |L_i| < L_max then w_{i+1} = w_i + w_inc  else w_{i+1} = w_i
return x from (x, e) ∈ L_v where HW(e) is minimal
```

Line 1: $L_0 = \{(\mathbf{x}_0, \mathbf{e}_0)\}$, $\mathbf{x}_0, \mathbf{e}_0$ are zero-dimensional vectors.

Line 4: $$\mathbf{b} = \left(\mathbf{c}_{K_{i-1}}, \ldots, \mathbf{c}_{K_i}\right) \,||\, \left(\mathbf{x}_{i-1} B'_i + \left(\mathbf{c}_{k+N_{i-1}}, \ldots, \mathbf{c}_{k+N_i}\right)\right)$$

Line 5: $\text{max-wt} = \min(w_i - HW(\mathbf{e}_{i-1}), w_b)$

Line 6: **foreach** $e' \in \left\{\mathbf{v} \in \mathbb{F}_2^{n_i+k_i} \mid HW(\mathbf{v}) \leq \textit{max-wt}\right\}$ **do**

Line 7: Find $\mathbf{x}'$ s.t. $\mathbf{x}' G_i + \mathbf{b} = \mathbf{e}'$

Line 8: $$\mathbf{e}_{\text{new}} = \Big((\mathbf{e}_{i-1})_1, \ldots, (\mathbf{e}_{i-1})_{K_{i-1}}, \mathbf{e}'_1, \ldots, \mathbf{e}'_{k_i}, (\mathbf{e}_{i-1})_{K_{i-1}}, \ldots, (\mathbf{e}_{i-1})_{K_{i-1}+N_{i-1}}, \mathbf{e}'_{k_i}, \ldots, \mathbf{e}'_{k_i+n_i}\Big)$$

Line 9: Add $(\mathbf{x}_{i-1}||\mathbf{x}', \mathbf{e}_{\text{new}})$ to L_i

Line 10: **if** $|L_i| < L_{\max}$ **then** $w_{i+1} = w_i + w_{\text{inc}}$ **else** $w_{i+1} = w_i$

Line 11: **return** $\mathbf{x}$ *from* $(\mathbf{x}, \mathbf{e}) \in L_v$ *where* $HW(\mathbf{e})$ *is minimal*

Algorithm 3: List-decoding StGen codes [SG15]

Outline

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